

SYMPLECTIC MODULAR SYMBOLS

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ABSTRACT. Let K/\mathbb{Q} be a number field with euclidean ring of integers \mathcal{O} . Let Γ be a finite-index torsion-free subgroup of the symplectic group $Sp_{2n}(\mathcal{O})$, and let N be the cohomological dimension of Γ . We exhibit a finite, geometrically-defined spanning set of $H^N(\Gamma; \mathbb{Z})$ by generalizing the modular symbol algorithm of Ash and Rudolph for $SL_n(\mathcal{O})$.

1. INTRODUCTION

1.1. Let G be a semisimple algebraic group defined over \mathbb{Q} of \mathbb{Q} -rank ℓ , and let X be the associated symmetric space. Let $\Gamma \subset G(\mathbb{Q})$ be a torsion-free arithmetic subgroup. Then $H^*(\Gamma; \mathbb{Z}) = H^*(\Gamma \backslash X; \mathbb{Z})$, and this cohomology vanishes for $* > N$, where $N = \dim(X) - \ell$, the *cohomological dimension* of Γ .

The theory of modular symbols as formulated by Ash [2] constructs an explicit spanning set for $H^N(\Gamma; \mathbb{Z})$ as follows. Let \mathcal{B} be the Tits building associated to G [17]. By the Solomon-Tits theorem, \mathcal{B} has the homotopy type of a wedge of $(\ell - 1)$ -spheres, and thus $\tilde{H}_*(\mathcal{B}; \mathbb{Z})$ is nonzero only in dimension $\ell - 1$. Using the Borel-Serre compactification of the locally symmetric space $\Gamma \backslash X$, we may construct a map

$$(1) \quad \Phi: H_{\ell-1}(\mathcal{B}; \mathbb{Z}) \longrightarrow H^N(\Gamma; \mathbb{Z})$$

that is surjective (cf. §2). Since the left-hand side of (1) is generated by fundamental classes of apartments of \mathcal{B} , this provides a geometric spanning set for $H^N(\Gamma)$. These cohomology classes (or rather, their duals in homology) are called *modular symbols*.

1.2. The modular symbols provide a spanning set for $H^N(\Gamma; \mathbb{Z})$, but they do not provide a finite spanning set, a distinction that is important for applications. However, suppose K/\mathbb{Q} is a number field with euclidean ring of integers \mathcal{O} , and let $G(\mathbb{Q}) = SL_n(K)$ and $\Gamma \subset SL_n(\mathcal{O})$. Then in [6], Ash and Rudolph determine an explicit finite spanning set—the *unimodular symbols*—and present an algorithm to write a modular symbol as a sum of unimodular symbols (cf. §2.9). This algorithm, in conjunction with certain explicit cell complexes, can be used to compute the action of the Hecke operators on $H^N(\Gamma)$. In turn, through work of Ash, Pinch, and Taylor [5], Ash

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and McConnell [4], and van Geemen and Top [18], this has produced corroborative evidence for certain aspects of the “Langlands philosophy.” In particular, in the case of $\Gamma \subset SL_3(\mathbb{Z})$, many examples of representations of the absolute Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ have been found that appear to be associated to cohomology classes of Γ .

1.3. In this paper we solve the finiteness problem for the symplectic group: $G(\mathbb{Q}) = Sp_{2n}(K)$ and Γ of finite index in $Sp_{2n}(\mathcal{O})$, where \mathcal{O} is euclidean. We characterize a finite spanning set of $H^N(\Gamma; \mathbb{Z})$ and present an algorithm (Theorem 4.11) that generalizes the modular symbol algorithm of [6]. To do this, we prove a relation in the homology of the symplectic building (Theorem 3.16) (see Examples 3.9 and 3.10).

1.4. We conclude this introduction by discussing the arithmetic nature of $H^N(\Gamma)$ in the symplectic case, and by indicating possible applications of Theorem 4.11.

First, we consider the complex cohomology. It is known that for $n > 1$, the groups $H^N(\Gamma; \mathbb{C})$ do not contain cuspidal classes (that is, cohomology classes corresponding to cuspidal automorphic forms). Moreover, work of Schwermer [13, 15, 14] shows the following:

- For $n > 1$, there are Eisenstein cohomology classes in the top degree. These classes are constructed using Eisenstein series and characters attached to the maximal split torus of the minimal parabolic subgroup of G . For $n > 3$, these are all the Eisenstein classes appearing in the top degree.
- Furthermore, for Sp_4 , there are additional classes constructed using Eisenstein series and cuspidal classes for SL_2 .

Hence for complex coefficients these classes are either easy to understand arithmetically, or are better studied on a lower rank group using the classical theory of modular symbols.

Nevertheless, there is one possibility that might be of interest. In the case of Sp_6 , one cannot exclude the possibility that there is an Eisenstein cohomology class in the top degree associated to a cuspidal class for Sp_4 whose infinity type is not a discrete series representation. It might be interesting to show the existence of this class and to study its arithmetic nature using the results of this paper. (I am grateful to the referee for this suggestion.)

1.5. Next, let p be a prime and let \mathbb{F}_p be the corresponding finite field. Consider the mod p cohomology $H^N(\Gamma; \mathbb{F}_p)$, or more generally $H^N(\Gamma; M)$, where M is an \mathbb{F}_p -module with Γ -action. In this setting the situation is much less clear. For example, there may be classes that lift to torsion classes in integral cohomology, and thus are not associated to automorphic forms in any obvious manner. For discussion and examples of this, see [1] for $\Gamma \subset GL_3(\mathbb{Z})$, and [3, 7] for GL_n . One would like to know if such classes arise in the symplectic case, and if so if the corresponding Hecke eigenclasses are attached to Galois representations. The algorithm in Theorem 4.11, in conjunction with the cell complex described in [12], provides the means to explore this question for $H^4(\Gamma)$, where $\Gamma \subset Sp_4(\mathbb{Z})$.

1.6. Finally, in another article [10], we describe an algorithm to compute the Hecke action on $H^5(\Gamma)$, where Γ is a subgroup of $SL_4(\mathbb{Z})$. This cohomology group, whose degree is one less than the cohomological dimension, is within the cuspidal range. One would like to have a symplectic version of this algorithm that could compute the Hecke action on cuspidal classes in H^3 of subgroups of $Sp_4(\mathbb{Z})$ (i.e. on Siegel modular forms of weight 3). The algorithm described in this paper is an essential first step towards solving this problem.

1.7. Acknowledgments/Related work. This paper relies heavily on the results of [2] and [6]. We thank Avner Ash for much advice and encouragement.

In [12] the authors describe a deformation retract W of the symmetric space $Sp_4(\mathbb{R})/U(2)$ that may be used to compute $H^*(\Gamma; \mathbb{Z})$ where Γ is a finite index subgroup of $Sp_4(\mathbb{Z})$. The combinatorial data used to describe the cell decomposition of W , found in [11], inspired the results in Example 3.9 and led to Theorem 3.16. We thank Bob MacPherson and Mark McConnell for many conversations.

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2. MINIMAL MODULAR SYMBOLS

In this section we review results on minimal modular symbols. These are due to Ash and Rudolph [6] for $G = SL_n$ and Ash [2] for any semisimple \mathbb{Q} -group G . Our exposition closely follows these sources. For general results about buildings, the reader may consult Tits [17].

2.1. Let G , X , and Γ be as in the introduction, and let $e \in X$ be the distinguished basepoint. Let T be a maximal \mathbb{Q} -split torus of G stable under the Cartan involution corresponding to e , and let $A = T(\mathbb{R})^0$. As ℓ is the \mathbb{Q} -rank of G , we have that $A \cong (\mathbb{R}^{>0})^\ell$.

Let \bar{X} be the partial compactification of X constructed by Borel and Serre [8]. Then the closure Z of Ae in \bar{X} is homeomorphic to a closed ball of dimension ℓ , and ∂Z lies in $\partial \bar{X}$. Let $[I] \in H_{\ell-1}(\partial \bar{X})$ be the fundamental class of ∂Z , and if $m \in G(\mathbb{Q})$, let $[m]$ be the fundamental class of $\partial(mZ)$.

Let \mathcal{B} be the Tits building associated to G . According to [8, §8.4.3], there is a homotopy equivalence $h: \mathcal{B} \rightarrow \partial \bar{X}$ that takes a distinguished apartment A_0 homeomorphically onto ∂Z . This map is compatible with the natural $G(\mathbb{Q})$ -action on \mathcal{B} and $\partial \bar{X}$. In particular, if $m \in G(\mathbb{Q})$, then $h(mA_0) = \partial(mZ)$. Since $G(\mathbb{Q})$ acts transitively on apartments, and $H_{\ell-1}(\mathcal{B}; \mathbb{Z})$ is generated by the fundamental classes of the apartments, we have shown the following:

2.2. Lemma. [2] *The classes $[m]$ generate $H_{\ell-1}(\partial \bar{X}; \mathbb{Z})$.*

2.3. Now let $\pi: \bar{X} \rightarrow \Gamma \backslash \bar{X}$ be the projection. Using π and the long exact sequence of the pair $(\bar{X}, \partial \bar{X})$, we obtain

$$(2) \quad H_{\ell-1}(\partial \bar{X}) \xrightarrow{\sim} H_{\ell}(\bar{X}, \partial \bar{X}) \xrightarrow{\pi_*} H_{\ell}(\Gamma \backslash \bar{X}, \partial(\Gamma \backslash \bar{X})) \xrightarrow{D} H^N(\Gamma \backslash \bar{X}) \xrightarrow{\sim} H^N(\Gamma \backslash X).$$

Here the first isomorphism follows from the contractibility of \bar{X} , the last isomorphism is induced by the canonical homotopy equivalence $X \hookrightarrow \bar{X}$, and the map D is Lefschetz duality. Since Γ is assumed torsion-free, D is an isomorphism with integral coefficients, and $H^N(\Gamma \backslash X; \mathbb{Z})$ can be identified with $H^N(\Gamma; \mathbb{Z})$. Let $[m]_{\Gamma}^* \in H^N(\Gamma; \mathbb{Z})$ be the image of $[m]$ under the sequence (2). One of the main results of [2] is that π_* is surjective, which with Lemma 2.2 implies

2.4. Theorem. [2] *As m varies over $G(\mathbb{Q})$, the classes $[m]_{\Gamma}^*$ generate $H^N(\Gamma; \mathbb{Z})$.*

2.5. Let K/\mathbb{Q} be a number field with ring of integers \mathcal{O} , and let G be the algebraic \mathbb{Q} -group such that $G(\mathbb{Q}) = SL_n(K)$. Here $\ell = n - 1$. We want to describe the classes $[m]$ using the combinatorics of \mathcal{B} .

Let $V = K^n$. We assume that the basis $\{e_1, \dots, e_n\} \subset V$ are eigenvectors for the torus T from §2.1. Then \mathcal{B} is a simplicial complex with a vertex for every proper nonzero subspace F of V . A set $\{F_1, \dots, F_k\}$ of vertices of \mathcal{B} spans a $(k-1)$ -simplex in \mathcal{B} if and only if the corresponding subspaces can be arranged in a proper flag:

$$0 \subset F_1 \subset \dots \subset F_k \subset V.$$

The action of $SL_n(K)$ on V induces an action on \mathcal{B} and on $H_*(\mathcal{B}; \mathbb{Z})$.

To construct the classes $[m]$, we use an auxiliary simplicial complex. Let $\llbracket n \rrbracket$ be the set $\{1, \dots, n\}$. Let $\partial \Delta_{n-1}$ be the barycentric subdivision of the boundary complex of the $(n-1)$ -simplex. In other words, $\partial \Delta_{n-1}$ is a simplicial complex with vertices corresponding to the proper non-empty subsets I of $\llbracket n \rrbracket$, and a collection of subsets I_1, \dots, I_k corresponds to a $(k-1)$ -simplex in $\partial \Delta_{n-1}$ if and only if they can be arranged in a proper flag: $I_1 \subset \dots \subset I_k$. We may orient $\partial \Delta_{n-1}$ using the standard ordering on $\llbracket n \rrbracket$.

Following [6], given n points in $V \setminus \{0\}$, we may construct a class in $H_{n-2}(\mathcal{B}; \mathbb{Z})$ as follows. Given $v_1, \dots, v_n \in V \setminus \{0\}$, we define a simplicial map

$$(3) \quad \phi: \partial \Delta_{n-1} \longrightarrow \mathcal{B}$$

by sending the vertex $I \subset \llbracket n \rrbracket$ to the vertex of \mathcal{B} corresponding to the subspace spanned by $\{v_i \mid i \in I\}$. If ξ is the fundamental class of the geometric realization of $\partial \Delta_{n-1}$, then $\phi_*(\xi)$ is a class in $H_{n-2}(\mathcal{B}; \mathbb{Z})$. Thus under the composition

$$\partial \Delta_{n-1} \xrightarrow{\phi} \mathcal{B} \xrightarrow{h} \partial \bar{X},$$

we have constructed a class in $H_{n-2}(\partial \bar{X}; \mathbb{Z})$.

2.6. Definition. [6] The *modular symbol* associated to $v_1, \dots, v_n \in V \setminus \{0\}$ is the class in $H_{n-2}(\partial \bar{X}; \mathbb{Z})$ constructed above, and is denoted $[v_1, \dots, v_n]$. If $m \in M_n(K)$

($n \times n$ matrices over K), then by $[m]$ we mean the modular symbol constructed using the columns of m .

2.7. Proposition. *If $m \in SL_n(K)$, then the construction of $[m]$ given in Definition 2.6 coincides with that given in §2.1. In particular, the classes $[m]$ span $H_{n-2}(\partial\bar{X}; \mathbb{Z})$.*

Proof. Recall that we have a homotopy equivalence $h: \mathcal{B} \rightarrow \partial\bar{X}$ taking a distinguished apartment A_0 homeomorphically onto ∂Z (§2.1). This apartment is in fact $\phi(\partial\Delta_{n-2})$, where $\partial\Delta_{n-2}$ is constructed using the basis $e_1, \dots, e_n \in V$. These identifications are compatible with the action of $SL_n(K)$ on V , \mathcal{B} , and $\partial\bar{X}$, so the result follows. \square

Modular symbols have the following properties.

2.8. Proposition. [6] *The map $[\]: M_n(K) \rightarrow H_{n-2}(\partial\bar{X}; \mathbb{Z})$ satisfies the following:*

1. $[v_1, \dots, v_n] = (-1)^{|\tau|} [\tau(v_1), \dots, \tau(v_n)]$, where $\tau \in S_n$ is a permutation on n letters, and $|\tau|$ is the length of τ .
2. If $q \in K$, then $[qv_1, v_2, \dots, v_n] = [v_1, \dots, v_n]$.
3. If the v_i are linearly dependent, then $[v_1, \dots, v_n] = 0$.
4. If $v_0, \dots, v_n \in V \setminus \{0\}$, then

$$\sum_i (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n] = 0.$$

Furthermore, $[\]$ is surjective.

2.9. Now assume that \mathcal{O} is a euclidean ring with respect to a multiplicative norm $\| \cdot \|: \mathcal{O} \rightarrow \mathbb{Z}^{\geq 0}$. Using multiplicativity extend the norm to a map $\| \cdot \|: K \rightarrow \mathbb{Q}^{\geq 0}$. We recall how to identify a Γ -finite spanning set of modular symbols. By a *primitive* vector, we mean a vector $v \in \mathcal{O}^n$ such that the greatest common divisor of the entries of v is a unit.

Let L be an \mathcal{O} -submodule of \mathcal{O}^n of rank $k \leq n$. Since \mathcal{O} is a principal ideal domain, L has a free \mathcal{O} -basis $B = \{v_1, \dots, v_k\}$. Choose $W = \{w_{k+1}, \dots, w_n\} \subset \mathcal{O}^n$ such that $B \sqcup W$ is a K -basis of K^n , and W may be extended to an \mathcal{O} -basis of \mathcal{O}^n .

We define the *index* of L by

$$i(L) := \|\det(v_1, \dots, v_k, w_{k+1}, \dots, w_n)\|.$$

It is easy to see that $i(L)$ is independent of the choices above, and that

$$i(L) = 1 \quad \text{if and only if} \quad (L \otimes_{\mathcal{O}} K) \cap \mathcal{O}^n = L.$$

Furthermore, if L has rank n and $\| \cdot \|$ is the usual norm $N_{K/\mathbb{Q}}: K \rightarrow \mathbb{Q}$, then $i(L) = [\mathcal{O}^n : L]$.

We write $i(v_1, \dots, v_k)$ for $i(L)$ if we are given a specific set of linearly independent vectors generating L . If the v_i are linearly dependent, we define $i(v_1, \dots, v_k) = 0$.

2.10. Definition. Let $v_1, \dots, v_k \in \mathcal{O}^n$ be linearly independent primitive vectors. Then a *candidate* for the v_i is a primitive $x \in \mathcal{O}^n \setminus \{0\}$ satisfying

$$0 \leq i(x, v_1, \dots, \hat{v}_i, \dots, v_k) < i(v_1, \dots, v_k), \quad 1 \leq i \leq k.$$

A fundamental result is the following:

2.11. Proposition. [6] *Let v_1, \dots, v_k be a linearly independent set of primitive vectors. If $i(v_1, \dots, v_k) > 1$, then a candidate x for the v_i exists.*

Proof. We give the proof since our statement differs slightly from that found in [6]. First assume $k = n$. Let L be the lattice spanned by the v_i , and let w be a primitive vector in \mathcal{O}^n that is not in L . Note that w exists since $i(L) > 1$.

Let $A \in M_n(\mathcal{O})$ be the matrix with columns v_1, \dots, v_n , and let $A_i[w]$ be the matrix obtained by replacing the i th column of A with w . Since \mathcal{O} is euclidean, there exist $\alpha_i, \beta_i \in \mathcal{O}$ such that

$$\det A_i[w] = \alpha_i \det A + \beta_i \quad \text{where } 0 \leq \|\beta_i\| < \|\det A\|.$$

Now let $x = w - \sum_i \alpha_i v_i$. By our choice of w the vector x is nonzero. It is easy to check that $\det A_i[x] = \beta_i$. Since $0 \leq \|\beta_i\| < \|\det A\|$ and $\|\det A_i[x]\| = i(x, v_1, \dots, \hat{v}_i, \dots, v_n)$, the result follows.

Now assume $k < n$. Since \mathcal{O} is euclidean, L is a free module. Hence we can choose an isomorphism $L \otimes K \rightarrow K^k$ that carries $(L \otimes K) \cap \mathcal{O}^n$ onto \mathcal{O}^k . This brings us back to the full rank setting, and we may argue as above. \square

Notice that x can be written as $\sum q_i v_i$, with $q_i \in K$ satisfying $0 \leq \|q_i\| < 1$. Furthermore, since the v_i are linearly independent, at least one $q_i \neq 0$.

2.12. Theorem. [6] *As m ranges over $SL_n(\mathcal{O})$, the classes $[m]$ generate $H_{n-2}(\partial \bar{X}; \mathbb{Z})$. Hence if $\Gamma \subset SL_n(\mathcal{O})$ is torsion-free and of finite index, then the classes $[m]_\Gamma^*$ provide a finite spanning set of $H^N(\Gamma; \mathbb{Z})$.*

Proof. Repeatedly applying Proposition 2.11 and Proposition 2.8 (4), we may write any class $[m]$ as a sum $\sum [m_i]$, where the determinant of each m_i is a unit. Then applying Proposition 2.8 (2), we may divide the first column of m_i by the determinant of $\det m_i$ to get $m'_i \in SL_n(\mathcal{O})$ satisfying $[m_i] = [m'_i]$. \square

2.13. Definition. The classes $\{[m] \mid m \in SL_n(\mathcal{O})\}$ are called *unimodular symbols*.

3. SYMPLECTIC MODULAR SYMBOLS

In this section we generalize the results of §2.5 to the symplectic case. Sections 3.1 and 3.2 recall well-known facts about symplectic geometry and the building associated to the symplectic group. In §3.4 we translate results of §2.5 to the symplectic setting, and in §§3.11-3.15 we describe a relation in the homology of the building that is crucial to our finiteness result. Throughout this section we do not assume that the ground field K has a euclidean ring of integers.

3.1. First we recall some elementary facts about symplectic geometry to fix notation. Fix a field K , and let V be the vector space K^{2n} . If $k \in \mathbb{Z}$, we use the notation \bar{k} for $2n + 1 - k$. We fix a non-degenerate alternating bilinear form $\langle \cdot, \cdot \rangle: V \rightarrow K$. A basis $v_1, \dots, v_n, v_{\bar{n}}, \dots, v_{\bar{1}}$ of V is said to be *symplectic* if

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } j = \bar{i} \text{ and } i < j, \\ -1 & \text{if } j = \bar{i} \text{ and } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

We let $G = Sp_{2n}(K)$ be the subgroup of $GL_{2n}(K)$ preserving the form. Thus $Sp_{2n}(K) = \{g \in GL_{2n}(K) \mid \langle gv, gw \rangle = \langle v, w \rangle\}$. The group G has \mathbb{Q} -rank $\ell = n$.

Given any $x \in V$, we define x^\perp to be the set $\{y \in V \mid \langle x, y \rangle = 0\}$. If x is nonzero, then x^\perp is a hyperplane containing x . A subspace $F \subset V$ is called *isotropic* if $\langle v, w \rangle = 0$ for all $v, w \in F$. Any one-dimensional subspace is isotropic, and the largest dimension an isotropic subspace may have is n . An n -dimensional isotropic subspace is called a *Lagrangian subspace*.

3.2. From now on we exclusively use \mathcal{B} to denote the building associated to $Sp_{2n}(K)$. We have the following description of \mathcal{B} as a simplicial complex, analogous to that found in §2.5: vertices of \mathcal{B} correspond to nonzero isotropic subspaces of V , and simplices of \mathcal{B} correspond to flags of nonzero isotropic subspaces.

To describe the geometry of \mathcal{B} we must use cross-polytopes instead of simplices, and so we recall their definition. Let e_1, \dots, e_n be the standard basis of \mathbb{R}^n . Then a *cross-polytope* on $2n$ vertices is a polytope isomorphic to the convex hull of the points $\pm e_1, \dots, \pm e_n$. Examples are the square ($n = 2$) and the octahedron ($n = 3$).

Since the proper faces of cross-polytopes are simplicial, we may regard their boundary complexes as simplicial complexes. Let $\partial\beta_n$ be the first barycentric subdivision of the boundary complex of the cross-polytope on $2n$ vertices. To describe the structure of $\partial\beta_n$, we use the notation $\llbracket n \rrbracket^\pm := \{1, \dots, n, \bar{n}, \dots, \bar{1}\}$. We order this set by $1 < \dots < n < \bar{n} < \dots < \bar{1}$.

3.3. Definition. (cf. [9]) A nonempty subset $I \subset \llbracket n \rrbracket^\pm$ is called *isotropic* if for all $i, j \in I$, we have $i \neq \bar{j}$.

Note that $\#I \leq n$ for any isotropic I . As in the SL_n case, vertices of $\partial\beta_n$ correspond to isotropic subsets of $\llbracket n \rrbracket^\pm$: if we take β to be the convex hull of the points $\pm e_1, \dots, \pm e_n$, then the set I corresponds to the linear span of $\{e_i \mid i \in I\}$, where $e_{\bar{i}} := -e_i$. Similarly, simplices of $\partial\beta_n$ correspond to proper flags of isotropic subsets.

3.4. Now suppose that we are given $2n$ points $v_1, \dots, v_n, v_{\bar{n}}, \dots, v_{\bar{1}} \in V \setminus \{0\}$.

3.5. Definition. The set $v_1, \dots, v_n, v_{\bar{n}}, \dots, v_{\bar{1}}$ is said to satisfy the *isotropy condition* if for every isotropic subset $I \subset \llbracket n \rrbracket^\pm$, the subspace spanned by $\{v_i \mid i \in I\}$ is isotropic.

Notice that this condition is strictly weaker than requiring that the v_i form a symplectic basis. In particular, the columns of any $m \in Sp_{2n}(K)$ satisfy this condition. Following §2.5, we define a simplicial map

$$\phi: \partial\beta_n \longrightarrow \mathcal{B}$$

by sending the vertex corresponding to an isotropic $I \subset \llbracket n \rrbracket^\pm$ to the vertex of \mathcal{B} corresponding to the linear span of $\{v_i \mid i \in I\}$. Via the composition $\phi \circ h$, we have an induced map on homology taking the fundamental class ξ of the geometric realization of $\partial\beta_n$ to a class in $H_{n-1}(\partial\bar{X}; \mathbb{Z})$.

3.6. Definition. Given $v_1, \dots, v_n, v_{\bar{n}}, \dots, v_{\bar{1}} \in V \setminus \{0\}$ satisfying the isotropy condition, let $[v_1, \dots, v_n; v_{\bar{n}}, \dots, v_{\bar{1}}] \in H_{n-1}(\partial\bar{X}; \mathbb{Z})$ denote the class constructed above. This class is called a *symplectic modular symbol*. If the columns of $m \in M_{2n}(K)$ satisfy the isotropy condition, we denote the symplectic modular symbol corresponding to its columns by $[m]$.

Symplectic modular symbols satisfy properties similar to those satisfied by the special linear symbols. For instance, minor modification of the proof of Proposition 2.7 shows that the construction of $[m]$ in Definition 3.6 agrees with that of §2.1, and thus the modular symbols span $H_{n-1}(\partial\bar{X}; \mathbb{Z})$. Furthermore, we have the following analog of the first parts of Proposition 2.8, whose proof is easily checked:

3.7. Proposition. *Symplectic modular symbols enjoy the following properties:*

1a. *Let $\tau \in S_n$ be a permutation on n letters. Then*

$$[v_1, \dots, v_n; v_{\bar{n}}, \dots, v_{\bar{1}}] = [\tau(v_1), \dots, \tau(v_n); \tau(v_{\bar{n}}), \dots, \tau(v_{\bar{1}})],$$

where $\tau(v_k) := v_{\tau(k)}$ and $\tau(v_{\bar{k}}) := v_{\overline{\tau(k)}}$ for $k \in \llbracket n \rrbracket$.

1b. $[v_1, v_2, \dots, v_n; v_{\bar{n}}, \dots, v_{\bar{2}}, v_{\bar{1}}] = -[v_{\bar{1}}, v_2, \dots, v_n; v_{\bar{n}}, \dots, v_{\bar{2}}, v_1]$.

2. *If $q \in K$, then $[qv_1, v_2, \dots, v_n; v_{\bar{n}}, \dots, v_{\bar{1}}] = [v_1, \dots, v_n; v_{\bar{n}}, \dots, v_{\bar{1}}]$.*

3. *If the v_i are linearly dependent, then $[v_1, \dots, v_n; v_{\bar{n}}, \dots, v_{\bar{1}}] = 0$.*

3.8. The symplectic analogue of Proposition 2.8 (4) is more complicated and forms one of the main results of this article. To motivate the result we illustrate it for $Sp_4(K)$ and $Sp_6(K)$. In light of the homotopy equivalence $\mathcal{B} \rightarrow \partial\bar{X}$ (§2.1), we work in $H_*(\mathcal{B})$ rather than $H_*(\partial\bar{X})$.

3.9. Example. Let $G = Sp_4(K)$. Then the simplicial complex \mathcal{B} is a graph with two types of vertices, corresponding to the one- and two-dimensional isotropic subspaces of V .

Let $[m] = [v_1, v_2; v_{\bar{2}}, v_{\bar{1}}]$ be a symplectic modular symbol for $Sp_4(K)$. We may picture the subspace configuration in $V = K^4$ determined by $\{v_i\}$ by passing to $\mathbb{P}^3(K)$. Figure 1 shows the projectivized configuration on the left, along with the apartment in \mathcal{B} corresponding to $[m]$ on the right. In \mathcal{B} we represent the one-dimensional (respectively two-dimensional) isotropic subspaces of V by solid (resp. hollow) vertices. By abuse of notation we use the same symbol for a point in K^4 , the

point it determines in $\mathbb{P}^3(K)$, and the vertex it determines in \mathcal{B} . The lines on the left of Figure 1 are the projectivizations of the Lagrangian planes determined by the v_i ; we have not drawn the images of the two non-Lagrangian planes.

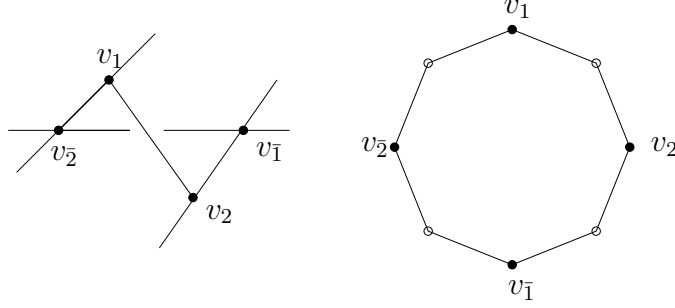


FIGURE 1. A configuration in $\mathbb{P}^3(K)$ and its corresponding apartment.

Now choose $x \in V \setminus \{0\}$. We will use x to construct a class in $H_1(\mathcal{B}; \mathbb{Z})$ homologous to $[m]$. Recall that x^\perp is the set of all $y \in V$ satisfying $\langle x, y \rangle = 0$. In $\mathbb{P}^3(K)$, x^\perp becomes a hyperplane that, for generic x , determines four new points by intersection with the original Lagrangian lines (see Figure 2). These new points and

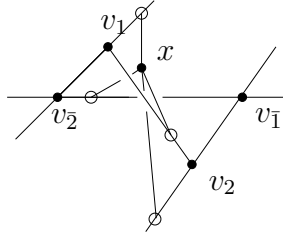


FIGURE 2. Constructing new points.

lines determine four new apartments in \mathcal{B} , and provide a relation $[m] = \sum_{i=1}^4 [m_i]$ in homology, as in Figure 3.

Now suppose that x is not generic. Then for some subset $I \subset \llbracket 2 \rrbracket^\pm$, we have $\langle x, v_i \rangle = 0$ for $i \in I$. Up to symmetry there are three nontrivial possibilities, which we depict in Figure 4. The corresponding apartments appear in Figure 5. Note that in these cases there are fewer apartments in the images of the relations.

3.10. Example. Consider the case $G = Sp_6(K)$. Now \mathcal{B} has three kinds of vertices, corresponding to isotropic points, lines, and planes in $\mathbb{P}^5(K)$. The configuration in $\mathbb{P}^5(K)$ corresponding to a symplectic modular symbol m is combinatorially an octahedron with isotropic faces. Choosing a generic point x , we construct twelve new points by intersecting x^\perp with the isotropic lines of the octahedron. In Figure 6 we show the configuration corresponding to $[m]$ along with these constructed points, which

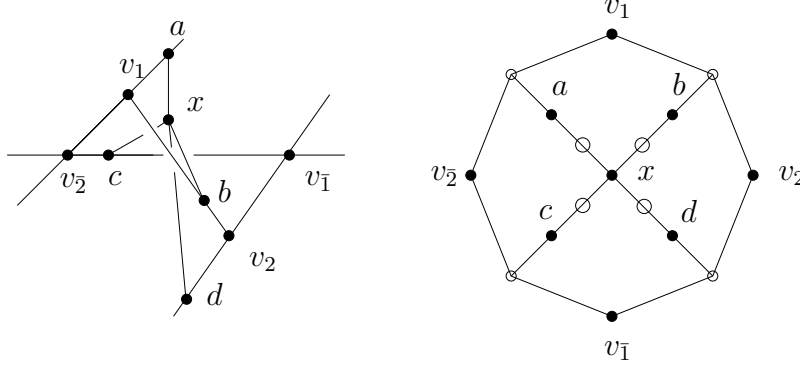


FIGURE 3. A homology relation.

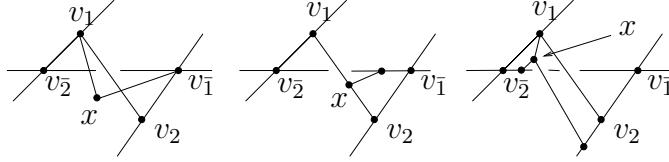
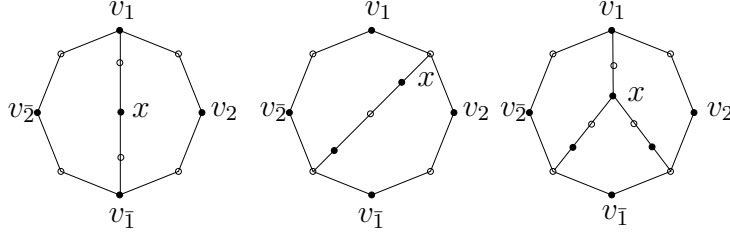
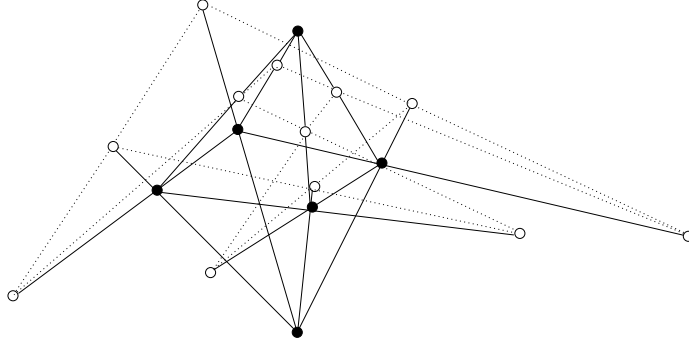
FIGURE 4. $I = \{1, \bar{1}\}, \{1, 2\}$, and $\{1\}$.

FIGURE 5.

are shown as hollow dots. (Although the configuration properly lives in $\mathbb{P}^5(K)$, we show it in three dimensions for clarity.) Notice that the constructed points satisfy nontrivial linear dependencies: three constructed points are collinear if they lie on an isotropic plane corresponding to a facet of the original octahedron. (These dependencies are only true in $\mathbb{P}^5(K)$.) These dependencies ensure the relation $[m] = \sum [m_i]$ in homology.

3.11. Returning to the general case, let $[m] = [v_1, \dots, v_n; v_{\bar{n}}, \dots, v_{\bar{1}}]$ be a symplectic modular symbol. Let $x \in V \setminus \{0\}$, and let x^\perp be as above. Let $D_x \subset \llbracket n \rrbracket^\pm$ be the set of indices such that $\langle x, v_i \rangle = 0$ if and only if $i \in D_x$. Given distinct $i, j \in \llbracket n \rrbracket^\pm$ with $i \neq \bar{j}$ and not both $i, j \in D_x$, define x_{ij} by

$$(4) \quad x_{ij} = \langle x, v_i \rangle v_j - \langle x, v_j \rangle v_i.$$


 FIGURE 6. A configuration in $\mathbb{P}^5(K)$.

If both i and j lie in D_x , then x_{ij} is not defined. The point x_{ij} lies on the intersection of x^\perp with the isotropic plane spanned by v_i and v_j .

Now we define new matrices built from m , x , and the x_{ij} .

3.12. Definition. Let $[m]$ be a symplectic modular symbol. Choose $x \in V \setminus \{0\}$, and construct the x_{ij} as in (4). If $i \notin D_x$, define the matrix m_i to be the matrix obtained by altering m according to the following rules:

1. Replace $v_{\bar{i}}$ by x .
2. For $j \in \llbracket n \rrbracket^\pm \setminus \{i, \bar{i}\}$, replace v_j by x_{ij} .

A priori $[m_i]$ may not be a symplectic modular symbol, since its columns might not satisfy the isotropy condition. Hence we state

3.13. Proposition. *For each $i \in \llbracket n \rrbracket^\pm \setminus D_x$, each $[m_i]$ is a symplectic modular symbol.*

Proof. It is easy to check that $\langle x_{ij}, x_{ik} \rangle = 0$ and $\langle v_i, x_{ij} \rangle = 0$ in all the necessary cases. Since $\langle x, x_{ij} \rangle = 0$ by construction, the result follows. \square

As indicated in Example 3.10, the x_{ij} satisfy linear dependencies, which we record in the following lemma.

3.14. Lemma. *Suppose $I = \{i, j, k\}$ is an isotropic subset, and $\#(I \cap D_x) \leq 1$. Then the points x_{ij} , x_{jk} , and x_{ik} are linearly dependent.*

Proof. The points satisfy the identity

$$\langle x, v_k \rangle x_{ij} = \langle x, v_j \rangle x_{ik} - \langle x, v_i \rangle x_{jk}.$$

\square

3.15. We come now to the main result of this section.

3.16. Theorem. *Let $[m] = [v_1, \dots, v_n; v_{\bar{n}}, \dots, v_{\bar{1}}]$ be a symplectic modular symbol for $Sp_{2n}(K)$, and choose $x \in V \setminus \{0\}$. Define D_x as above, and let $[m_i]$ be the*

symplectic modular symbols constructed in Definition 3.12. Then in $H_{n-1}(\partial\bar{X}; \mathbb{Z})$, we have

$$(5) \quad [m] = \sum_{i \in \llbracket n \rrbracket^\pm \setminus D_x} [m_i].$$

Proof. As in Examples 3.9 and 3.10, we prove the relation in $H_{n-1}(\mathcal{B}; \mathbb{Z})$. Let A (respectively A_i) be the apartment corresponding to the symplectic modular symbol $[m]$ (respectively $[m_i]$). We think of these apartments as being explicit simplicial cycles in \mathcal{B} , and we will show that $[m] = \sum_i [m_i]$ by examining these cycles.

We begin by fixing some notation. Let (a_1, \dots, a_k) be an ordered tuple of linearly independent points of V lying in a Lagrangian subspace, and let $F(a_1, \dots, a_k) \subset V$ be their linear span. Let $\sigma(a_1, \dots, a_k) \subset \mathcal{B}$ be the simplex corresponding to the flag

$$0 \subset F(a_1) \subset F(a_1, a_2) \subset \dots \subset F(a_1, \dots, a_k) \subset V.$$

Recall that the (*closed*) *star* of a simplex σ in a simplicial complex is the set of all simplices σ' meeting σ , as well as the faces of all such σ' . Also recall that maximal simplices in \mathcal{B} are called *chambers*.

First assume that the point x is generic with respect to the v_i , so that $D_x = \emptyset$. Consider the column vector v_i from $[m]$. The chambers in A appearing in the star of v_i are the simplices of the form

$$(6) \quad \sigma(v_i, v_{k_1}, \dots, v_{k_{n-1}}) \subset A,$$

where $\{i, k_1, \dots, k_{n-1}\} \subset \llbracket n \rrbracket^\pm$ is isotropic. These chambers correspond to the flags

$$(7) \quad 0 \subset F(v_i) \subset F(v_i, v_{k_1}) \subset \dots \subset F(v_i, v_{k_1}, \dots, v_{k_{n-1}}) \subset V.$$

On the right of (5), in A_i , we have the chambers

$$(8) \quad \sigma(v_i, x_{i,k_1}, \dots, x_{i,k_{n-1}}) \subset A_i.$$

These chambers correspond to the flags

$$(9) \quad 0 \subset F(v_i) \subset F(v_i, x_{i,k_1}) \subset \dots \subset F(v_i, x_{i,k_1}, \dots, x_{i,k_{n-1}}) \subset V.$$

The sets of flags in (7) and (9) coincide by the definition of the x_{ij} , and these chambers appear with the same orientations on both sides of (5). Taking all permutations of $\{k_1, \dots, k_{n-1}\}$ in (6)–(9), we obtain all chambers in the star of v_i . Hence any chamber in A appears once in a unique A_i with the same orientation.

Now we claim that each of the remaining chambers in the A_i appears exactly twice with opposite orientations. Any such chamber must appear in the star of x or x_{ij} , and we first consider the star of x . Choose an apartment A_i , a point x_{ij} , and let $I \subset \llbracket n \rrbracket^\pm$ be a maximal isotropic subset of the form $\{i, j, k_1, \dots, k_{n-2}\}$. Let F_I be the Lagrangian subspace corresponding to I . Consider the chambers with x and x_{ij} as vertices, and with all vertices other than x corresponding to subspaces lying in F_I . These chambers have the form

$$(10) \quad \sigma(x, x_{ij}, x_{i,k_1}, \dots, x_{i,k_{n-2}}) \subset A_i$$

or

$$(11) \quad \sigma(x, x_{ij}, x_{j,k_1}, \dots, x_{j,k_{n-2}}) \subset A_j.$$

In (10) and (11), we allow all permutations of $\{k_1, \dots, k_{n-2}\}$. By Lemma 3.14, these chambers correspond to the same isotropic flag. Furthermore, it is not difficult to see that the chambers in (10) and (11) appear in A_i and A_j with opposite orientations. We may apply this argument to any pair $\{i, j\}$ and any F_I with $\{i, j\} \subset I$, and so all the chambers in the star of x in the A_i cancel each other in (5).

To complete the proof for generic x , we investigate any remaining chambers on the right-hand side of (5). These must appear in the stars of the x_{ij} . Fix x_{ij} , and let $I = \{i, j, k_1, \dots, k_{n-2}\}$ and F_I be as above. The chambers meeting the star of x_{ij} and with all vertices corresponding to subspaces lying in F_I are of two types: first,

$$(12) \quad \sigma(x_{ij}, x_{i,k_1}, \dots, x_{i,k_{n-2}}, v_i) \subset A_i \quad \text{and}$$

$$(13) \quad \sigma(x_{ij}, x_{j,k_1}, \dots, x_{j,k_{n-2}}, v_j) \subset A_j,$$

and secondly

$$(14) \quad \sigma(x_{ij}, x_{i,k_1}, \dots, x_{i,k_{n-2}}, x) \subset A_i \quad \text{and}$$

$$(15) \quad \sigma(x_{ij}, x_{j,k_1}, \dots, x_{j,k_{n-2}}, x) \subset A_j.$$

In (12)–(15), we allow all permutations of the right $n-1$ vertices. By Lemma 3.14, the isotropic flags corresponding to (12) and (13) (respectively (14) and (15)) coincide, and checking orientations shows that these cancel in pairs. This accounts for all the chambers on both sides of (5), and hence the result follows for generic x .

Now assume that $D_x \neq \emptyset$. Write $D_x = I \sqcup J$, where $I = \bar{I}$ and $J \cap \bar{J} = \emptyset$. We claim it is sufficient to assume $I = \emptyset$. Indeed, let $S = \llbracket n \rrbracket^\pm \setminus I$, and let $V' \subset V$ be the span of $\{v_i \mid i \in S\}$. Then $x \in V'$, and V' is a symplectic space whose form $\langle \ , \ \rangle'$ is the restriction of $\langle \ , \ \rangle$. Furthermore, the vectors $\{v_i \mid i \in S\}$ define an apartment in the building \mathcal{B}' associated to $(V', \langle \ , \ \rangle')$, and thus determine a symplectic modular symbol $[m'] \in H_{n'-1}(\mathcal{B}'; \mathbb{Z})$, where $n' = \frac{1}{2} \# S$. Writing $[m] = \sum [m_i]$ in $H_{n-1}(\mathcal{B}; \mathbb{Z})$ is equivalent to writing $[m'] = \sum [m'_i]$ in $H_{n'-1}(\mathcal{B}'; \mathbb{Z})$. Hence, by induction we may assume that D_x contains no subset I with $\bar{I} = I$.

This is the same as D_x being isotropic, and so up to symmetry the only invariant of D_x is its cardinality. As before we proceed by investigating the stars of vertices. We merely indicate which chambers cancel which and omit the details.

We first consider the case that $\#D_x \leq n-1$. Any chamber in the star of v_i has the form

$$(16) \quad \sigma(v_i, v_{k_1}, \dots, v_{k_{n-1}}) \subset A,$$

where $\{i, k_1, \dots, k_{n-1}\} \subset \llbracket n \rrbracket^\pm$ is isotropic. This is matched on the right side of (5) by the chamber

$$(17) \quad \sigma(v_i, v_{k_1}, \dots, v_{k_j}, x_{k_j, k_{j+1}}, \dots, x_{k_j, k_{n-1}}) \subset A_{k_j}.$$

In (17), k_j is the first subscript reading from the left that doesn't appear in D_x . (The possibility that $i = k_j$ is included.) This chamber appears with the same orientation as that of (16), and chambers of this form account for all chambers on the left side of (5).

In the star of x , the chamber

$$(18) \quad \sigma(x, x_{ij}, x_{i,k_1}, \dots, x_{i,k_{n-2}}) \subset A_i$$

is canceled by the chamber

$$(19) \quad \sigma(x, x_{ij}, x_{j,k_1}, \dots, x_{j,k_{n-2}}) \subset A_j,$$

exactly as in the generic case. If $\#D_x < n - 1$, then every chamber in the star of x is of this form up to symmetry. If $\#D_x = n - 1$, the remaining chambers in the star of x have the form

$$(20) \quad \sigma(x, v_{i_1}, \dots, v_{i_{n-1}}),$$

where $D_x = \{i_1, \dots, i_{n-1}\}$. This chamber will appear in A_k and $A_{\bar{k}}$ with opposite orientations, where k and \bar{k} are the unique elements that extend D_x to an isotropic subset.

The remaining chambers on the right side of (5) appear in the stars of the x_{ij} where $\{i, j\} \not\subset D_x$. These cancel exactly as in (12)–(15).

Finally we consider the case $\#D_x = n$. This case is slightly different, since x is in the span of the $\{v_i \mid i \in D_x\}$. Again we begin with the star of the v_i , and consider the chamber

$$\sigma(v_i, v_{k_1}, \dots, v_{k_{n-1}}).$$

If $D_x \neq \{i, k_1, \dots, k_{n-2}\}$, then this chamber is matched by the chamber

$$(21) \quad \sigma(v_i, v_{k_1}, \dots, v_{k_j}, x_{k_j, k_{j+1}}, \dots, x_{k_j, k_{n-1}}) \subset A_{k_j}.$$

exactly as in (17). Otherwise, if $D_x = \{i, k_1, \dots, k_{n-2}\}$, then this chamber is matched by

$$\sigma(v_i, v_{k_1}, \dots, v_{k_{n-2}}, x) \subset A_{\overline{k_{n-1}}}.$$

Now consider the star of x . If $\{i, j\} \cap D_x = \emptyset$, then we have cancellation as in (18) and (19). Otherwise, write $D_x = \{i_1, \dots, i_n\}$. Then in the remaining chambers,

$$\sigma(x, v_{i_1}, \dots, v_{i_{n-1}}) \subset A_{\overline{i_n}}$$

cancels

$$\sigma(x, v_{i_1}, \dots, v_{i_{n-2}}, v_{i_n}) \subset A_{\overline{i_{n-1}}}.$$

Finally, the chambers in the star of the x_{ij} with $\{i, j\} \not\subset D_x$ cancel exactly as in (12)–(15). □

3.17. *Remark.* Theorem 3.16 can be proven in more generality than stated here. For example, the proof applies to buildings associated to the odd orthogonal groups SO_{2n+1} , and can be modified to work for buildings associated to the even orthogonal groups SO_{2n} . Using the notion of a “perspectivity” [16], we may prove a similar result for buildings of type G_2 . We would like to have a “building-theoretic” proof of Theorem 3.16.

4. FINITENESS

Throughout this section we assume that \mathcal{O} is a euclidean ring with respect to the norm $\|\cdot\|: \mathcal{O} \rightarrow \mathbb{Z}^{\geq 0}$. We also assume, using Proposition 3.7 (2), that all modular symbols have integral columns.

4.1. If $m \in Sp_{2n}(\mathcal{O})$, we call the class $[m] \in H_{n-1}(\partial\bar{X}; \mathbb{Z})$ a *unimodular symbol*. The goal of this section is to prove that the unimodular symbols span $H_{n-1}(\partial\bar{X}; \mathbb{Z})$. We begin with a notion that lets us measure how far a symplectic modular symbol is from being unimodular. In contrast to the special linear case (§2.9), we use the symplectic pairing rather than the determinant as a measure of non-unimodularity.

4.2. Definition. Let $[m]$ be a symplectic modular symbol with primitive columns. The *depth* of $[m] = [v_1, \dots, v_n; v_{\bar{n}}, \dots, v_{\bar{1}}]$ is the number

$$d(m) := \max_{i \in \llbracket n \rrbracket} \left\{ \|\langle v_i, v_{\bar{i}} \rangle\| \right\}.$$

Notice that $d(m) = 1$ implies that each $\langle v_i, v_{\bar{i}} \rangle \in \mathcal{O}^\times$. Hence if $d(m) = 1$ we may divide the columns of m by appropriate units to obtain $m' \in Sp_{2n}(\mathcal{O})$ satisfying $[m] = [m']$. Thus to show that the unimodular symbols span $H_{n-1}(\partial\bar{X}; \mathbb{Z})$, it is sufficient to show that through a homology we may replace a modular symbol with depth > 1 with a cycle of modular symbols, each of which has smaller depth.

4.3. Lemma. *Let $[m] = [v_1, \dots, v_n; v_{\bar{n}}, \dots, v_{\bar{1}}]$ be a symplectic modular symbol with primitive columns. If $d(m) > 1$, then $i(v_1, \dots, v_n; v_{\bar{n}}, \dots, v_{\bar{1}}) > 1$, and there exists a candidate for the v_i .*

Proof. Assume that $i(v_1, \dots, v_n; v_{\bar{n}}, \dots, v_{\bar{1}}) = 1$. Then the lattice generated by the v_i is \mathcal{O}^n , and thus $m \in GL_{2n}(\mathcal{O})$. Therefore

$$\det(m) = \prod_{i \in \llbracket n \rrbracket} \langle v_i, v_{\bar{i}} \rangle \in \mathcal{O}^\times,$$

and $d(m) = 1$, a contradiction. This implies $i(v_1, \dots, v_n; v_{\bar{n}}, \dots, v_{\bar{1}}) > 1$, and a candidate exists by Proposition 2.11. \square

4.4. Suppose that $[m] = [v_1, \dots, v_n, v_{\bar{n}}, \dots, v_{\bar{1}}]$ is a symplectic modular symbol with $d(m) > 1$, and let x be a candidate for m . As in §2.9, write

$$x = \sum_{i \in \llbracket n \rrbracket^\pm} q_i v_i$$

with $q_i \in K$ satisfying $0 \leq \|q_i\| < 1$. Define $[m_i]$ as in Definition 3.12, so that

$$(22) \quad [m] = \sum_{i \in \llbracket n \rrbracket^\pm \setminus D_x} [m_i].$$

Notice that for $i \in \llbracket n \rrbracket^\pm \setminus D_x$, we have

$$\|\langle x, v_i \rangle\| = \|q_i \langle v_i, v_i \rangle\| < \|\langle v_i, v_i \rangle\| \leq d(m),$$

so that the norm of at least one of the symplectic pairings in each m_i has decreased. However, it is not true that $d(m_i) < d(m)$ in general, and so more than just the construction of x is required to show that the unimodular symbols span. The following example illustrates our strategy.

4.5. Example. Consider the case of $G = Sp_6(K)$. As in Example 3.10, $[m]$ corresponds to an octahedron in $\mathbb{P}^5(K)$. Suppose that x is a candidate for $[m]$, and that $D_x = \emptyset$. Construct the modular symbols $[m_i]$. Then the configuration in $\mathbb{P}^5(K)$ corresponding to $[m_1]$ is the octahedron with vertices v_1, x , and the four constructed points lying on the lines in the original configuration in the link of v_1 . (See Figure 7.)

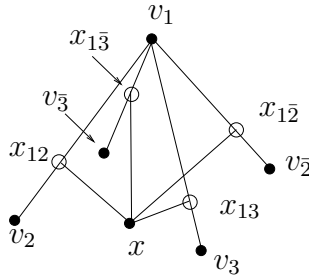


FIGURE 7. The configuration corresponding to $[m_1]$.

The point x has been chosen so that $\|\langle x, v_1 \rangle\| < d(m)$, but in general $\|\langle x_{12}, x_{1\bar{2}} \rangle\|$ and $\|\langle x_{13}, x_{1\bar{3}} \rangle\|$ will be larger than $d(m)$. However, we claim that $i(x_{12}, x_{13}, x_{1\bar{3}}, x_{1\bar{2}}) > 1$, and that we may identify the quadruple $(x_{12}, x_{13}, x_{1\bar{3}}, x_{1\bar{2}})$ with a modular symbol $[m']$ for $Sp_4(K)$. This means we may argue inductively and use the reduction of this “link” modular symbol to reduce $[m_1]$. (See Figure 8.)

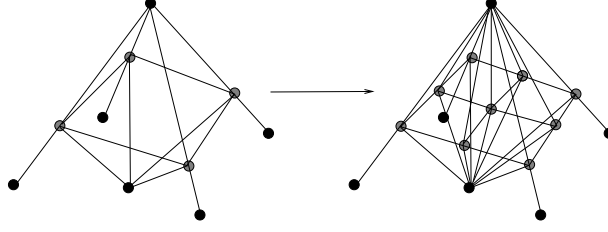


FIGURE 8. Reducing the link modular symbol.

4.6. To study the $[m_i]$, we first prove a version of Hermite normal form for our matrix representatives.

4.7. Lemma. *Let $m \in M_{2n}(\mathcal{O})$ be a matrix with nonzero, primitive columns, and assume that m satisfies the isotropy condition. Then there is a matrix $\gamma \in Sp_{2n}(\mathcal{O})$ such that γm is upper triangular and γm satisfies the isotropy condition.*

Proof. The left action of $Sp_{2n}(\mathcal{O})$ corresponds to row operations on m , so we begin by listing the elementary symplectic row operations. Let $\{r_i \mid i \in \llbracket n \rrbracket^\pm\}$ denote the rows of m , and use the notation $a \leftarrow b$ to mean that the row vector a is to be replaced by the expression b . Then we find that we may effect the following:

- (T1) $r_i \leftarrow r_i + r_{\bar{i}}$
 (T2) for $1 \leq i < k \leq n$,

$$\begin{aligned} r_i &\leftarrow r_i + r_k \\ r_{\bar{k}} &\leftarrow r_{\bar{k}} - r_{\bar{i}} \end{aligned}$$

- (T3) for $1 \leq i < k \leq n$,

$$\begin{aligned} r_i &\leftarrow r_i + r_{\bar{k}} \\ r_k &\leftarrow r_k + r_{\bar{i}} \end{aligned}$$

- (P1)

$$\begin{aligned} r_i &\leftarrow r_{\bar{i}} \\ r_{\bar{i}} &\leftarrow -r_i \end{aligned}$$

- (P2) if $\tau \in S_n$, then

$$r_i \leftarrow r_{\tau(i)},$$

Here T stands for *transvection* and P stands for *permutation*. In (P2) we mean that τ permutes the first n rows among themselves, with the action on the last n rows determined by $\tau(\bar{i}) := \overline{\tau(i)}$. Furthermore, the inverses and transpositions of these operations are also elementary row operations.

Now, using operations of type (T1) and (P1), we may carry m into the following form:

$$\begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1\bar{1}} \\ \vdots & \vdots & & \vdots \\ m_{n,1} & m_{n,2} & \cdots & m_{n,\bar{1}} \\ 0 & m_{\bar{n},2} & \cdots & m_{\bar{n},\bar{1}} \\ \vdots & \vdots & & \vdots \\ 0 & m_{\bar{1}2} & \cdots & m_{\bar{1}\bar{1}} \end{pmatrix}.$$

In the first column, the entry in row i is the greatest common divisor of the original entries in row i and row \bar{i} . Next, using operations of type (T2) and (P2), we can take m into the form

$$\begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1\bar{1}} \\ 0 & m_{22} & \cdots & m_{2\bar{1}} \\ \vdots & \vdots & & \vdots \\ 0 & m_{\bar{1}2} & \cdots & m_{\bar{1}\bar{1}} \end{pmatrix}.$$

Since left multiplication by $\gamma \in Sp_{2n}(\mathcal{O})$ preserves the isotropy condition, it follows that m actually has the form

$$\begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1\bar{1}} \\ 0 & m_{22} & \cdots & m_{2\bar{1}} \\ \vdots & \vdots & & \vdots \\ \vdots & m_{\bar{2}2} & \cdots & m_{\bar{2}\bar{1}} \\ 0 & \cdots & 0 & m_{\bar{1}\bar{1}} \end{pmatrix}.$$

Now we may apply the induction hypothesis to the middle $(2n-2) \times (2n-2)$ block to complete the proof of the lemma. \square

4.8. Let $[m]$ be a symplectic modular symbol with m upper triangular, and let x be a candidate for m . Without loss of generality, we may assume $v_1 = e_1$ by Proposition 3.7 (3), and that $1 \notin D_x$. We want to study the modular symbol $[m_1]$ from (22) in greater detail.

4.9. Lemma. *Suppose that m is upper triangular with $v_1 = e_1$. Let the points*

$$\{x_{1,j} \mid j \in \llbracket n \rrbracket^\pm \setminus \{1, \bar{1}\}\}$$

and the matrix m_1 be defined as in Definition 3.12. Let X be the $2n \times (2n-2)$ matrix with columns the vectors

$$x_{12}, \dots, x_{1,n}, x_{1,\bar{n}}, \dots, x_{1\bar{2}}.$$

That is, X is the matrix obtained by deleting the first and last columns from m_1 . Define points w_j for $j \in \llbracket n \rrbracket^\pm \setminus \{1, \bar{1}\}$ by

$$w_j = \langle e_j, x \rangle e_1 - \langle e_1, x \rangle e_j,$$

and let W be the $2n \times (2n - 2)$ matrix with columns the vectors

$$w_2, \dots, w_n, w_{\bar{n}}, \dots, w_{\bar{2}}.$$

Let m' be the central $(2n - 2) \times (2n - 2)$ block of m_1 . Then

$$X = Wm'.$$

Proof. Given a column vector v , let v^k be the entry in the k -th row. Suppose first that $j \leq n$. Then computing using Definition 3.12, we find

$$x_{1,j}^1 = x^{\bar{2}}m_{2,j} + \dots + x^{\bar{j}}m_{jj},$$

and

$$x_{1,j}^k = -x^{\bar{1}}m_{kj} \quad \text{for } 2 \leq k \leq j.$$

On the other hand, W has the form

$$\begin{pmatrix} x^{\bar{2}} & x^{\bar{3}} & \dots & x^2 \\ -x^{\bar{1}} & 0 & \dots & 0 \\ 0 & -x^{\bar{1}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -x^{\bar{1}} \end{pmatrix},$$

and so $x_{1,j}$ is this matrix times the j -th column of m' . The computation is similar if $j > n$, so the result follows. \square

4.10. We come now to the main result of this article.

4.11. Theorem. *As m ranges over $Sp_{2n}(\mathcal{O})$, the classes $[m]$ span $H_{n-1}(\partial\bar{X}; \mathbb{Z})$.*

Proof. By the paragraph following Definition 4.2, it is sufficient to show that if $[m]$ satisfies $d(m) > 1$, then $[m] = \sum [m_\alpha]$, where $d(m_\alpha) < d(m)$. We proceed by induction. Since $SL_2(\mathcal{O}) = Sp_2(\mathcal{O})$, the statement is true by the usual modular symbol algorithm. So we assume the statement is true for $Sp_{2k}(\mathcal{O})$ with $k < n$.

Assume that $d(m) > 1$, and let x be a candidate for m . Write

$$(23) \quad [m] = \sum_{i \in \llbracket n \rrbracket^\pm \setminus D_x} [m_i].$$

We permute the columns of each $[m_i]$ so that v_i is the first column.

Choose an $[m_i]$ from the right of (23). Permuting labels if necessary, we can assume $i = 1$. Since candidate selection and the relation (5) are $Sp_{2n}(\mathcal{O})$ -equivariant, using Lemma 4.7 we may assume m_1 is upper triangular. As in Lemma 4.9, we construct the matrix m' and the vectors w_j . Since m' is a $(2n - 2) \times (2n - 2)$ matrix satisfying the isotropy condition, we have that $[m']$ corresponds to a class in $H_{n-2}(\mathcal{B}'; \mathbb{Z})$, where \mathcal{B}' is the building associated to $Sp_{2n-2}(K)$. By the induction hypothesis we may write

$$(24) \quad [m'] = \sum_{\alpha \in A} [m'_\alpha],$$

where $m'_\alpha \in Sp_{2n-2}(\mathcal{O})$, and the sum is finite.

We may use the $[m'_\alpha]$ to write

$$(25) \quad [m_1] = \sum_{\alpha \in A} [m_\alpha]$$

as follows. Each m'_α corresponds to an endomorphism of the \mathcal{O} -module generated by the w_j . We may apply m'_α to W to produce a $2n \times (2n-2)$ matrix W_α . Then m_α is the matrix with first column e_1 , last column x , and with middle columns W_α . The induction hypothesis asserts that the columns of W_α form an \mathcal{O} -basis of the \mathcal{O} -module generated by the w_j . In particular,

$$d(m_\alpha) \leq \|\langle w_j, w_{\bar{j}} \rangle\| = \|\langle x, v_1 \rangle\|^2.$$

We claim that we may reduce each $[m_\alpha]$ further to write

$$[m_\alpha] = \sum_{\beta \in B} [m_{\alpha\beta}],$$

where

$$d(m_{\alpha\beta}) \leq \|\langle x, v_1 \rangle\|.$$

To see this, consider the j and \bar{j} columns of W :

$$\begin{aligned} w_j &= (x^{\bar{j}}, 0, \dots, 0, -x^{\bar{1}}, 0, \dots, 0)^t \quad \text{and} \\ w_{\bar{j}} &= (-x^j, 0, \dots, 0, -x^{\bar{1}}, 0, \dots, 0)^t, \end{aligned}$$

where $-x^{\bar{1}}$ appears in the j -th and \bar{j} -th rows respectively. Assume first that these vectors are primitive. Then some multiple of w_j added to $w_{\bar{j}}$ will be divisible by $x^{\bar{1}}$. This means w_j and $w_{\bar{j}}$ generate a lattice L satisfying

$$i(L) \geq \|x^{\bar{1}}\| = \|\langle x, v_1 \rangle\|.$$

Therefore we may apply the SL_2 -modular symbol algorithm to the pair $(w_j, w_{\bar{j}})$ to reduce L to $(L \otimes K) \cap \mathcal{O}^n$. If either w_j or $w_{\bar{j}}$ is not primitive, then a simple modification of this argument achieves the same end.

Applying this to each pair $(w_j, w_{\bar{j}})$, and adapting the construction that passes from (24) to (25), we find

$$(26) \quad [m_\alpha] = \sum_{\beta \in B} [m_{\alpha\beta}],$$

with

$$d(m_{\alpha\beta}) \leq \|\langle x, v_1 \rangle\|.$$

Together (25) and (26) imply

$$[m_1] = \sum_{\substack{\alpha \in A \\ \beta \in B}} [m_{\alpha\beta}] \quad \text{with } d(m_{\alpha\beta}) \leq \langle x, v_1 \rangle < d(m).$$

Since this argument may be applied to any of the $[m_i]$ from (23), this completes the proof of the theorem. \square

4.12. Corollary. *If $\Gamma \subset Sp_{2n}(\mathcal{O})$ is torsion-free of finite index, and N is the cohomological dimension of Γ , then the classes $[m]_{\Gamma}^*$ provide a finite spanning set of $H^N(\Gamma; \mathbb{Z})$.*

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